J. Fluid Mech. (1965), *vol.* 21, *part* 2, *pp.* 211–223
Printed in Great Britain

A three-dimensional model of inertial currents in a variable-density ocean

By A. R. ROBINSON
Pierce Hall, Harvard University

(Received 4 June 1964)

The three-dimensional equations for a general inertial ocean current are transformed so that the temperature rather than the vertical co-ordinate appears as an independent variable. A downstream power series expansion is made of the equations and boundary conditions, which involves an expansion about the mean potential vorticity. A general first-order solution is obtained for a boundary current between two level surfaces, one of no motion and one of uniform temperature. The case of constant potential vorticity is treated for arbitrary inviscid boundary conditions; it is found that the current can exist as a boundary layer only if the open ocean geostrophic drift is westward everywhere in the depth interval. This result is extended to arbitrary potential vorticity distributions by an asymptotic analysis in physical space.

1. The inertial ocean current

1.1. *Introduction*

Inertial currents and jets are an important and isolatable feature of the theory of the general circulation of the oceans (e.g. Stommel 1958; Greenspan 1963; Robinson 1963). By such a current is meant a flow in which the vertical component of vorticity relative to the rotating earth is comparable to the planetary vorticity (twice the vertical component of diurnal rotation). Since the current is a narrow region of relatively intense flow, the relative vorticity is dominated by the cross-stream gradient of the downstream velocity component. Consistently, the divergence of advective flux is important only in the cross-stream momentum balance. The downstream component is in geostrophic balance, but the relevant pressure gradient is much larger than that characteristic of the geostrophic drift of the fluid surrounding the current. Since the latter gradient is characteristic of the overall oceanic driving forces, the current is a region of free flow, driven by a mass flux through its lateral boundaries.

Previous studies of inertial currents have been essentially of a two-dimensional nature, and density variation has been modelled in a two-layer approximation. In the present study a model appropriate to the three-dimensional flow in an ocean of continuously variable density field is postulated. It is assumed that the Boussinesq approximation is valid and that the motion is adiabatic, i.e. the density field is merely advected in the current region. A mathematical framework is evolved for the study of general boundary and free jets in which the effects of bottom topography are included. A particular problem is then studied in detail: the effect of density gradients upon the boundary current with N–S axis in the presence of a variable Coriolis parameter (β-effect). It is hoped that the present model will not only provide a basis for the problems mentioned here, but may also be logically extended to provide a systematic framework for the investigation of inertial jet instability and the exploration of momentum and vorticity transfer with the solid earth.

1.2. *Equations of the model*

Let (x, y, z) represent respectively the cross-stream, downstream, and vertical direction. A non-dimensional density anomaly is defined by $\rho = \rho_0(1-s)$, whence

$$s = 1-\rho/\rho_0 = a(T^*-T_0^*), \tag{1.1}$$

in terms of the apparent temperature $T^*-T_0^* = (T-T_0)-(b/a)(S-S_0)$ (where a is the coefficient of thermal, and b of haline, expansion). Under standard notation, the conservation of momentum, mass and density flux is expressed, consistently with the assumptions discussed in the preceding section, by

$$-fv+\frac{1}{\rho_0}p_x = 0, \tag{1.2}$$

$$uv_x+vv_y+wv_z+fu+\frac{1}{\rho_0}p_y = 0, \tag{1.3}$$

$$-sg+\frac{1}{\rho_0}p_z = 0, \tag{1.4}$$

$$u_x+v_y+w_z = 0, \tag{1.5}$$

$$us_x+vs_y+ws_z = 0. \tag{1.6}$$

Under the β-plane approximation the Coriolis parameter is expressed by

$$f(x,y) = f_0+\beta(\cos\Theta y-\sin\Theta x), \tag{1.7}$$

where Θ is the clockwise-positive angle of deviation of the mean jet axis from the northward direction. Usually, but not always, the cross-stream variation must be neglected for consistency with the scale-ratio expansion implicit in the form of equations (1.2) and (1.3).

To determine the inertial current the following boundary conditions must be specified: the normal velocity and density distributions at the inlet and at the sides, as well as the kinematical conditions of vanishing normal velocity at the sea bottom and surface. Let $y = 0$ be the inlet; then it is required to specify

$$v(x,0,z), \quad s(x,0,z). \tag{1.8}$$

If the jet is free, i.e. if both the right- and left-hand regions of the current are open ocean regions, then we must require that

$$u \to U^{\pm\infty}(y,z), \quad v, w \to 0, \quad s \to S^{\pm\infty}(y,z) \quad \text{as} \quad x \to \pm\infty. \tag{1.9a}$$

Furthermore, the asymptotic normal flow must be constrained to satisfy (1.3) in the geostrophic form. Together with (1.4) this implies the thermal wind equations,

$$fU_z^{\pm\infty}+gS_y^{\pm\infty} = 0, \tag{1.9b}$$

so that U and S may not be chosen completely independently. In terms of boundary-layer analysis, condition (1.9*b*) accomplishes a smooth joining of the boundary-layer flow (the inertial jet) to the flow outside of the singular region (the open ocean). If, however, the jet is not entirely free but is bounded on one side (at $x = C(y)$) by a continental land mass then one of condition (1.9*a*) is replaced by

$$u(C(y), y, z) - v(C, y, z)\, C'(y) = 0. \tag{1.10}$$

The sea bottom is taken to lie at $z = B(x,y)$ and the sea surface, assumed undistorted, at $z = H$. Then

$$-w(x,y,B) + u(x,y,B)\, B_x + v(x,y,B)\, B_y = w(x,y,H) = 0. \tag{1.11}$$

This completes the statement of the problems to be considered.

2. Formulation of the problem in temperature space

2.1. *Transformation to a quasi-Lagrangian co-ordinate*

A direct attack on the mathematical problem posed above is almost prohibitively difficult, especially because of the non-linear form of the three-dimensional advection of momentum and density. However, if the stratification is everywhere stable (s_z a monotonic, non-vanishing function of z) equation (1.6) suggests that the conservation equations will have simpler form if horizontal differentiation is carried out along surfaces of constant density anomaly (Starr 1945). Let (x, y, s) be independent variables and z a dependent variable. Then equations (1.2)–(1.6) transform to

$$-fv + \Pi_x = 0, \tag{2.1}$$

$$uv_x + vv_y + fu + \Pi_y = 0, \tag{2.2}$$

$$gz + \Pi_s = 0, \tag{2.3}$$

$$(uz_s)_x + (vz_s)_y = 0, \tag{2.4}$$

$$-w + uz_x + vz_y = 0, \tag{2.5}$$

where subscripts now refer to partial differentiation in the new set, i.e. $u_x = (\partial u/\partial x)_{y,s}$, etc. Here the function $\Pi = p/\rho_0 - gsz$ conveniently replaces the pressure as an independent variable.

The advantages of the temperature space co-ordinates are the two-dimensional form of the momentum advection in (2.2), and the appearance of the vertical velocity directly and only in (2.5) so that w may be found as a subsidiary calculation after (u, v, z, Π) are known. Mass-continuity appears non-linearly, but the two-dimensional form of (2.4) allows the definition of a stream-like function, viz.

$$uz_s = -\psi_y, \quad vz_s = \psi_x, \tag{2.6}$$

for the product of the horizontal velocity and the inverse stratification,

$$z_s = 1 \Big/ \left(\frac{\partial s}{\partial z}\right)_{x,y}.\dagger$$

A difficulty of the present system is the cumbersome non-linear form, taken on by boundary conditions at level surfaces, e.g. the second of (1.11).

† In the two-layer models of Charney (1955) and Morgan (1956), the stream-like function involves the product of horizontal velocity and upper-layer depth, D, and is usually interpreted as a transport-function. But the role of D is also to define the strength of the stratification in a two-density model, and this interpretation is apparently more relevant here.

2.2. *First integrals of the motion* (*conservation of potential vorticity*)

The integral theorems of the two-layer models may be generalized to three dimensions, the relevant quantities being conserved along a path of constant values of both the stream-like function and the density. Cross-differentiation of (2.1), (2.2) and substitution of the horizontal divergence $u_x + v_y$ from (2.4) yields

$$uz_s\left(\frac{f+v_x}{z_s}\right)_x + vz_s\left(\frac{f+v_x}{z_s}\right)_y = 0.$$

Hence by (2.6)

$$v_x + f = z_s P(\psi, s), \tag{2.7}$$

where P is an arbitrary functional of its two arguments. Equation (2.7) is the conservation of potential vorticity.

A Bernoulli integral is obtained upon integration of the sum of (2.1) and (2.2) after respective multiplication by u, v, viz.

$$\tfrac{1}{2}v^2 + \Pi = B(\psi, s). \tag{2.8}$$

But if (2.8) is differentiated with respect to x at constant s, and the geostrophic equation (2.1) is applied, upon comparison with (2.7) it is found that

$$\partial B(\psi, s)/\partial\psi = P(\psi, s),$$

so that the potential vorticity and Bernoulli integrals (as in the two-layer case), are not independent. It will be convenient to employ (2.7), and (2.1) and to ignore (2.8).

3. Development of solutions for the complete current

3.1. *The downstream expansion*

Interest in the subject of inertial currents first arose when Stommel showed that a meaningful model of a Gulf-Stream region results from the assumption that the potential vorticity in a layer of variable depth is an absolute constant (Stommel 1958, p. 109; Robinson 1963, p. 154). The subsequent exploration of two-layer models has shown that inertial currents can exhibit a variety of interesting phenomena when the potential vorticity is a slowly varying function of its argument. This fact will be exploited into a formal expansion procedure about the mean value of the potential vorticity in the region of interest.

That such a procedure will facilitate solution can be seen by noting that if in (2.7) $P(\psi, s)$ is a function of s alone, (2.7) together with the geostrophic and hydrostatic equations (2.1), (2.3) provide three linear equations for the variables v, Π, z. The u, w fields can then be computed from (2.2), (2.5); but the forms which result contain in general products and ratios of infinite sums of fundamental separated solutions in the x, s co-ordinates, and also contain the co-ordinate y parametrically. To apply the boundary conditions (1.11) (and possible (1.10)) is prohibitively difficult. This difficulty can however be simply removed by the device of a power series expansion in the downstream co-ordinate, y. The type of downstream variation accessible to such an expansion can easily describe phenomena of interest.

It is convenient at this point to introduce non-dimensional variables and parameters. The flow is characterized by U_0 (a typical value of the entrainment or ejection rate $U^\infty(y,z)$), $\Delta\rho$ (a measure of the density difference from bottom to surface), H (the mean depth), and f_0 (the value of the Coriolis parameter at the origin). Let

$$\xi = \left(\frac{\rho_0 f_0^2}{\Delta\rho g H}\right)^{\frac{1}{2}} x, \quad \eta = \left(\frac{\rho_0 f_0 U_0}{\Delta\rho g H}\right) y, \quad \zeta = \frac{z}{H}, \quad \theta = \left(\frac{\rho_0}{\Delta\rho}\right) s. \tag{3.1}$$

Note that the jet width has been scaled by the radius of deformation rather than by the width scale appropriate to the two-dimensional jet, $(U_0/\beta)^{\frac{1}{2}}$. In the two-layer model these lengths are identical; here they are independent. The arbitrariness thus implied is reflected in the non-dimensional form of the Coriolis parameter,

$$\phi = \frac{f}{f_0} = 1 + \tilde{\beta}\xi + \beta^*\eta = \phi_0 + \beta^*\eta, \tag{3.2}$$

where $$\beta^* = \frac{\beta\cos\Theta\Delta\rho g H}{\rho_0 f_0^2 U_0}, \quad \tilde{\beta} = \frac{-\beta\sin\Theta}{f_0^2}\left(\frac{\Delta\rho g H}{\rho_0}\right)^{\frac{1}{2}}.$$

The remaining dependent variables u, v, w, Π, ψ are non-dimensionalized respectively by U_0, $(\Delta\rho g H/\rho_0)^{\frac{1}{2}}$, $U_0 f_0(\rho_0 H/\Delta\rho g)^{\frac{1}{2}}$, $\Delta\rho g H/\rho_0$, gH^2/f_0; the potential vorticity functional P, by $\Delta\rho f_0/\rho_0 H$. The symbols, u, v, w, Π, P will be retained for the non-dimensional variables (which will be recognized as such by their non-dimensional arguments). The choice of scaling is such that no parameters (other than β^*, $\tilde{\beta}$ in ϕ) appear in the non-dimensional forms of equations (2.1)–(2.7). The neglected inertial terms in (2.1) are simply $O(U_0^2\rho_0/\Delta\rho g H)$ compared to a retained term.

The downstream expansion is now made by writing all fields as power series in η, e.g.

$$\sum_{i=0}^{\infty} u_i \eta^i.$$

The result is

$$(2.1) \rightarrow \quad -\phi_0 v_0 + \Pi_{0\xi} = 0, \tag{3.3}$$

$$(2.2) \rightarrow \begin{cases} -\phi_0 v_1 + \beta^* v_0 + \Pi_{1\xi} = 0, & (3.4) \\ u_0 v_{0\xi} + v_0 v_1 + \phi_0 u_0 + \Pi_1 = 0, & (3.5) \end{cases}$$

$$(2.3) \rightarrow \begin{cases} \zeta_0 + \Pi_{0\theta} = 0, & (3.6) \\ \zeta_1 + \Pi_{1\theta} = 0, & (3.7) \end{cases}$$

$$(2.4) \rightarrow \quad (u_{0\xi} + v_1)\zeta_{0\theta} + u_0\zeta_{0\theta\xi} + v_0\zeta_{1\theta} = 0, \tag{3.8}$$

$$(2.5) \rightarrow \begin{cases} w_0 = u_0\zeta_{0\xi} + v_0\zeta_1, & (3.9) \\ w_1 = u_1\zeta_{0\xi} + u_0\zeta_{1\xi} + v_1\zeta_1 + 2v_0\zeta_2, & (3.10) \end{cases}$$

$$(2.6) \rightarrow \begin{cases} u_0\zeta_{0\theta} = -\psi_1, & (3.11) \\ v_0\zeta_{0\theta} = \psi_{0\xi}, & (3.12) \\ v_0\zeta_{1\theta} + v_1\zeta_{0\theta} = \psi_{1\xi}. & (3.13) \end{cases}$$

Physical consistency requires the simultaneous determination of expansion coefficients of different indices, as can be seen immediately from the form of the equation. The equations (3.3)–(3.13) will be of interest here. The potential vorticity integral expands as

$$P(\psi(\xi,\eta,\theta),\theta) = P(\psi(\xi,0,\theta),\theta)+P_\eta(\psi(\xi,0,\theta),\theta)\,\eta+\tfrac{1}{2}P_{\eta\eta}(\psi(\xi,0,\theta),\theta)\,\eta^2+\ldots;$$

but
$$P_\eta = P_\psi\psi_\eta,\quad P_{\eta\eta} = P_\psi\psi_{\eta\eta}+P_{\psi\psi}\psi_\eta^2,$$
whence

$$P = P_0(\psi_0,\theta)+P_I(\psi_0,\theta)\,\psi_1\eta+\{P_I(\psi_0,\theta)\,\psi_2+P_{II}(\psi_0^2,\theta)\,\psi_1^2\}\,\tfrac{1}{2}\eta^2+\ldots,$$

where the $P_N(\psi_0(\theta,\xi),\theta)$ are a set of independent functions defining the potential vorticity distribution of the jet. Thus

$$(2.7)\quad\rightarrow\quad\begin{cases} v_{0\xi}+\phi_0 = \zeta_0 P_0, & (3.14a)\\ v_{1\xi}+\beta^* = \zeta_{0\theta}P_I\psi_I+\zeta_{1\theta}P_0. & (3.14b)\end{cases}$$

Boundary conditions, e.g. the first of (1.11), require consistent transformation and expansion. $\theta_B(\xi,\eta) = \sum_{i=0}^{\infty} b_i(\xi)\,\eta^i$, the temperature at the surface

$$\zeta = B(\xi,\eta) = \sum_{i=0}^{\infty} B_i(\xi)\eta^i,$$

is implicitly defined by

$$\begin{aligned}\zeta(\xi,\eta,\theta_B(\xi,\eta)) &= \zeta_0(\xi,\theta_B)+\eta\zeta_1(\xi,\theta_B)+\ldots\\ &= \zeta_0(\xi,b_0(\xi))+\eta b_1\,\zeta_{0\theta}(\xi,b_0)+\eta\zeta_1(\xi,b_0)+O(\eta^2),\end{aligned}$$

whence
$$\zeta_0(\xi,b_0(\xi)) = B_0(\xi),\quad b_1(\xi)\,\zeta_{0\theta}(\xi,b_0)+\zeta_1(\xi,b_0) = B_1(\xi). \tag{3.15}$$

Then
$$-w(\xi,\eta,\theta_B(\xi,\eta))+u(B_0'+\eta B_1'+\ldots)+v(B_1+2\eta B_2+\ldots) = 0,$$

where u, v have the same argument as w, and a prime indicates total differentiation. Therefore

$$w_0(\xi,b_0(\xi)) = u_0(\xi,b_0)\,B_0'(\xi)+v_0(\xi,b_0)\,B_1(\xi), \tag{3.16a}$$

$$b_1(\xi)\,w_{0\theta}(\xi,b_0(\xi))+w_1(\xi,b_0) = u_0B_1'+(b_1u_{0\theta}+u_1)\,B_0'+2v_0B_2+(b_1v_{0\theta}+v_1)\,B_1 \tag{3.16b}$$

to $O(\eta^2)$.

3.2. *The boundary current with N–S axis*

To illustrate the general procedure, consider a three-dimensional form of the Gulf-Stream problem prevalent in ocean-circulation theory. Let $\Theta = 0$, $v_0 = w_0 = \psi_0 = 0$, and $P_N(0,\theta) = P_N^0(\theta)$. An initial calculation of $\zeta_0(\theta)$, $\Pi_0(\theta)$ may be made from (3.6) and (3.14a) in the form $\zeta_{0\theta}P_0^0(\theta) = 1$. The velocity field u_0, v_1, w_1 is determined simultaneously with ζ_1, Π_1 (and ψ_1) from (3.4), (3.5), (3.7), (3.10), (3.11), (3.14b). The pressure function Π_1 satisfies a linear second-order equation (in general with non-constant coefficients),

$$\Pi_{1\xi\xi}+P_0^0\,\Pi_{1\theta\theta}-\frac{P_1^0}{P_0^{02}}\,\Pi_1 = -\beta^*, \tag{3.17}$$

and the related fields are given directly by

$$u_0 = -\Pi_1,\quad v_1 = \Pi_{1\xi},\quad \zeta_1 = -\Pi_{1\theta},\quad \psi_1 = \Pi_1/P_0^0,\quad w_1 = \Pi_1\Pi_{1\theta\xi}-\Pi_{1\theta}\Pi_{1\xi}. \tag{3.18}$$

Note that to this order the cross-stream component u_0 is computed geostrophically, although the inherent non-linearity of the system is essential to produce the Π_1-equation (a quasi-geostrophic approximation in meteorological terminology). This occurs because $v_0 = 0$; u_1 is not computed geostrophically.

Consider a straight coastline at $\xi = 0$ and a flat bottom at $\zeta = 0$. By (3.18), (1.10) implies $\Pi_1(0, \theta) = 0$; (1.9) is to be satisfied by requiring $\Pi_1(\xi, \theta) \to \Pi_1^\infty(\theta)$ as $\xi \to \infty$, where Π_1^∞ is any solution of (3.17) with the first term absent. Equation (3.16*a*) is identically zero. Since $\zeta_{0\xi} = B_{0\xi} = 0$ the first of (3.15) implies $b_0(\xi) = b_{00}$ (a constant), whence (3.16*b*) by the last of (3.18) yields

$$\Pi_1 \Pi_{1\theta\xi} - \Pi_{1\theta} \Pi_{1\xi} = 0, \quad \text{when} \quad \theta = b_{00}. \tag{3.19}$$

The non-linear condition (3.19) may be integrated in ξ, i.e.

$$\ln \Pi_1 - \ln \Pi_{1\theta} = \text{const.},$$

or

$$\Pi_1(\xi, b_{00}) = k_0 \Pi_{1\theta}(\xi, b_{00}), \quad k_0 = \Pi_1^\infty(b_{00})/\Pi_{1\theta}^\infty(b_{00}), \tag{3.20a}$$

a linear form. An identical treatment of the second condition of (1.11) at $\zeta = 1$ yields

$$\Pi_1(\xi, h_{00}) = k_1 \Pi_1(\xi, h_{00}), \quad k_1 = \Pi_1^\infty(h_{00})/\Pi_{1\theta}^\infty(h_{00}), \tag{3.20b}$$

where $\zeta_0(h_{00}) = 1$.

Equation (3.17) and its boundary conditions allow separation of variables. Let

$$\Pi_1(\theta, \xi) = \Pi_1^\infty(\theta) + \sum_{n=0}^{\infty} a_n e^{-\lambda_n \xi} \Pi_{1n}(\theta),$$

where Π_1^∞ is any solution of

$$\Pi_1^{\infty\prime\prime} - (P_1^0/P_0^{03}) \Pi_1^\infty = -\beta^*/P_0^0, \tag{3.21}$$

and Π_{1n} satisfies

$$P_0^0 \Pi_{1n}'' + (\lambda_n^2 - P_1^0/P_0^{02}) \Pi_{1n} = 0, \tag{3.22a}$$

$$\Pi_{1n}(b_{00}) - k_0 \Pi_1'(b_{00}) = \Pi_{1n}(h_{00}) - k_1 \Pi_1(h_{00}) = 0, \tag{3.22b}$$

which serve to determine the eigenfunctions and eigenvalue spectrum λ_n. Then the a_n are fixed by the condition

$$\sum_{n=0}^{\infty} a_n \Pi_{1n}(\theta) = -\Pi_1^\infty(\theta). \tag{3.23}$$

The existence of the inertial boundary current requires real eigenvalues ($\lambda_n^2 > 0$ to ensure exponential decay at the seaward edge of the jet).†

In general all the eigenfunctions are required by (3.23) so that all the λ_n must be real. This condition selects a class of solutions of (3.21) which may couple with an inertial current to satisfy the requirement of no normal flow through a continental boundary. The determination of this class for a given set of P_N^0 is the first question of interest; the structure and properties of the various inertial jets required by the allowed distributions of geostrophic entrainment or ejection at the seaward edge, is the second. Two requirements govern our development: (1) the description of real oceanic jets, and (2) the understanding of such naturally occurring jets to be afforded by recognizing their place within a general framework of solutions.

† If the eigenvalue is complex then the condition is that the positive part be real. In a variety of examples treated to date only purely real or imaginary eigenvalues have arisen, but we have not proven a theorem.

$P_0^0(\theta)$ determines the basic stratification of the system, e.g. if a thermocline structure is to be described by an e-fold decay of temperature within a depth d of the surface, then $\theta = 1-e^{-\zeta_0/d}$ and $P_0^0 = 1/\zeta_{0\theta} = (1-\theta)/d$. On the other hand the thermoclined flow will contain certain features in common with flows which imply simpler versions of equations (3.21), (3.22), and these will be explored first. Thus let the P_N^0 be constants, i.e. $P_0^0 = 1$, $P_1^0 = \mp\beta^*$. The former choice implies a uniform basic stratification $\zeta_0 = \theta$; thus $b_{00} = 1$, $h_{00} = 0$. The choice of P_1^0 implies (without loss of generality) that the particular solution of (3.21) is a unit barotropic mode to the east or west ($U_1^\infty = \pm 1$).† Consider now the flow in which the horizontal velocity vanishes at the bottom, and the temperature remains constant at the top. Then $k_0 = 0$, $k_1 \to \infty$ and $\Pi_1(\xi, 0) = \Pi_{1\theta}(\xi, 1) = 0$. For $P_1^0 = -\beta^*$, the relevant solution of (3.21) is

$$\Pi_1^\infty = -U_1^\infty = -1+\cos\beta^{*\frac{1}{2}}(\theta-1)/\cos\beta^{*\frac{1}{2}}. \tag{3.24}$$

The appropriate complete set of eigenfunctions is $\Pi_1^n = \cos\frac{1}{2}n\pi(\theta-1)$, $n = 1, 3, 5, \ldots$. Upon substitution of this form, (3.22a) becomes the characteristic equation $\lambda_n^2 = (\frac{1}{2}n\pi)^2-\beta^*$. The lowest eigenvalue $\lambda_1^2 = \frac{1}{4}\pi^2-\beta^*$, so the condition for the existence of the inertial boundary current is $\beta^{*\frac{1}{2}} < \frac{1}{2}\pi$. From (3.24) it is seen that the allowed flows U_1^∞ are monotonic functions of the depth and are everywhere westward. Thus the two-dimensional condition of westward geostrophic drift (Carrier & Robinson 1962) obtains in the example. The complete solution for the jet velocity distribution is given to $O(\eta^2)$ by

$$\left.\begin{aligned} v &= -\eta\sum_{n=0}^{\infty}\lambda_n a_n\cos\frac{n\pi}{2}(\theta-1)\,e^{-\lambda_n\xi},\\ a_n &= \frac{16\beta^*}{\pi^3}\frac{(-)^n}{2n+1}\left[\frac{4\beta^*}{\pi^2}-(2n+1)^2\right].\end{aligned}\right\} \tag{3.25}$$

The case $P_1^0 = +\beta^*$ has also been solved; U_1^∞ is again everywhere westward and develops a boundary-layer behaviour near $\theta = 0$.

3.3. *Constant potential vorticity*

In this section an investigation is made into the general dependence of the boundary jet solutions upon the parameters k_0, k_1 which characterize the geostrophic drift. This will be done for the simplest choice of potential vorticity distribution, $P_0^0 = 1$, $P_{N>0}^0 = 0$; i.e. when $P(\psi, \theta)$ is an absolute constant. The solution of (3.21) is

$$\Pi_1^\infty = \beta^*\left[-\frac{\theta^2}{2}+\frac{\frac{1}{2}-k_1}{1+k_0-k_1}(\theta+k_0)\right]; \tag{3.26}$$

the particular solution is an eastward flow which increases quadratically from the bottom. Since k_0 and k_1 are arbitrary the flow may be unidirectional to the east or west or may change sign once or twice in the interval $(0, 1)$ of θ. Since (3.22a) has constant coefficients, the solution may be written

$$\Pi_{1n} = a_n\sin\lambda_n\theta+b_n\cos\lambda_n\theta.$$

† This flow may furthermore be regarded as the extension to three dimensions of the 'simple wind-system' flow discussed by Morgan (1956).

Application of the first of conditions (3.22*b*) at $\theta = 0$ yields $b_n = k_0 \lambda_n a_n$. From the second of (3.22*b*) at $\theta = 1$ there results the characteristic equation

$$\sin \lambda_n (1 + k_0 k_1 \lambda_n^2) = (k_1 - k_0) \lambda_n \cos \lambda_n. \tag{3.27}$$

The problem under consideration may be shown to be of Sturm–Liouville form with λ_n^2 as the eigenvalue. Therefore λ_n^2 is real and λ_n is either real or imaginary. To find restrictions upon k_0, k_1 to ensure the existence of the inertial boundary layer, it is sufficient to determine the sign of the lowest eigenvalue λ_0^2. Assume that $\lambda_0 = i\gamma$, γ real. Then by (3.27)

$$\tanh \gamma = \frac{(k_1 - k_0)\gamma}{1 - k_1 k_0 \gamma^2} = F(\gamma). \tag{3.28}$$

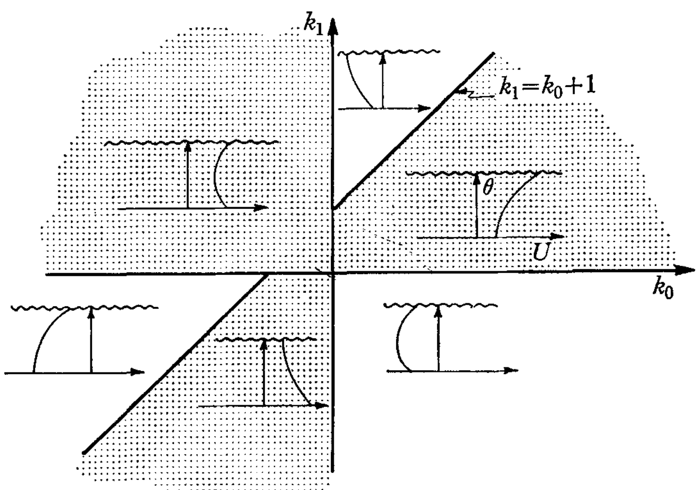

FIGURE 1. The (k_0, k_1)-plane. In the unshaded regions, a boundary layer can exist; in the shaded it cannot. Existence occurs along the axes except for $k_0 = 0$, $0 < k_1 < 1$ and $k_1 = 0$, $-1 < k_0 < 0$. Some profiles of the velocity U_1^∞ typical to the different regions are sketched, but various other profiles are possible (see, for example, (3.26) ff.).

Since (3.28) is unaltered when $\gamma \to -\gamma$, the values of k_0, k_1 for which $F(\gamma)$ intersects $\tanh \gamma$ may be sought for $\gamma \geqslant 0$. The result is shown in figure 1. In the first quadrant $k_0, k_1 > 0$ and F changes sign about the singularity at $\gamma_\infty = (k_0 k_1)^{-\frac{1}{2}}$. If $k_0 > k_1$, $F(\gamma > \gamma_\infty) > 0$ and F intersects $\tanh \gamma$. If $k_0 < k_1$, $F(\gamma > \gamma_\infty) < 0$ and an intersection will occur for $\gamma < \gamma_\infty$ only if $F(\gamma)$ rises less rapidly from zero than $\tanh \gamma$ does. Expansion for small γ,

$$\tanh \gamma = \gamma(1 - \tfrac{1}{3}\gamma^2 + \ldots), \quad F(\gamma) = (k_1 - k_0)\gamma(1 + k_0 k_1 \gamma^2 + \ldots),$$

shows that no intersection occurs for $k_1 - k_0 > 1$. In the second quadrant F is always positive and has a maximum at $\gamma_M = (-k_0 k_1)^{-\frac{1}{2}}$; $F(\gamma_M) = r + r^{-1}$, $r = (-k_1/k_0)^{\frac{1}{2}}$. Since $dF/dr = 0$ when $r = 1$, the minimum value of the maximum of F is 2. Thus F always intersects tanh which asymptotes to 1. The behaviour in the third quadrant is similar to the first; in the fourth quadrant $F < 0$ always.

The character of the solution (3.26) is illustrated by sketches of typical velocity profiles (U_∞^1 in the various regions of the (k_0, k_1)-plane, figure 1. The condition that an inertial jet be capable of allowing the geostrophic drift to satisfy the continental boundary condition is that the drift be everywhere westward. The condition is not upon the geostrophic transport; no part of the profile may be eastward. Any distribution of purely westward flow consistent with constant potential vorticity is allowed.

4. Interaction of boundary current with geostrophic drift

4.1. *Asymptotic equations for the seaward edge*

The requirement of westward flow obtained above is of particular interest when considering the role of inertial currents in the general ocean circulation. It is important to know if the condition holds for non-constant potential vorticity distributions. The question has been explored (Greenspan 1963) for a homogeneous density model with variable bottom depth, and for a two-layer model with the lower layer at rest. Here the effects of stratification alone will continue to be isolated by retaining $B(x,y) = 0$. However, the problem is fundamentally a non-linear one, and both mechanisms which may produce an 'effective-β', topography and stratification, must be explored simultaneously. This problem has been investigated and the results will be reported subsequently.

The development in §§2 and 3 above has been pointed towards obtaining solutions for the entire jet region. To inquire into the interaction of the current with the open ocean drift it is simpler to restrict attention to the seaward edge of the boundary current, i.e. to perform an asymptotic analysis in the boundary-layer variable. A linearization may be made about the flow at infinity. Dimensional dependent and independent variables and the (x,y,z) co-ordinate system will be used. Let $u = U(y,z)+\mu(x,y,z)$, $s = S(y,z)+\sigma(x,y,z)$ where the superscript ∞ has been dropped. Then since $\mu, v, w, \sigma \to 0$ as $x\to\infty$, these fields are small quantities at the boundary-layer edge and their squares may be neglected. From (1.2), (1.3), (1.5) an asymptotic vorticity equation is

$$Uv_{xx}+\beta v-fw_z = 0, \tag{4.1}$$

and by (1.2), (1.4), (1.6), (1.9*b*) the vertical velocity is

$$w = \frac{f}{gS_z}(U_z v - Uv_z). \tag{4.2}$$

Thus v satisfies

$$v_{zz}-\frac{S_{zz}}{S_z}v_z+\frac{S_z g}{f^2}v_{xx}+\frac{1}{U}\left[\frac{S_z g\beta}{f^2}-U_{zz}+\frac{S_{zz}}{S_z}U_z\right]v = 0. \tag{4.3a}$$

$$U_z v - Uv_z = 0 \quad \text{at} \quad z = 0, H. \tag{4.3b}$$

Note that the y-dependence of (4.3*a*) is parametric and that the coefficients are independent of x, whence the system is separable. Let

$$v(x,y,z) = \sum_n A_n(y)\, e^{-l_n(y)}v_n(z). \tag{4.4}$$

The coefficients A_n are in principle to be determined by a coupling of the asymptotic fields to a form of the solution valid inshore, but (4.3) serves to determine the eigenvalue spectrum l_n and thus solves the existence problem in general. Consider the case of quadratic drift and linear stratification of the open ocean. With $S_{zz} = 0$, $U = \mathscr{U}\zeta(1+\zeta/c)$, $\zeta = (z-z_0)/H$, (4.3) under (4.4) becomes

$$v_{n\zeta\zeta} + \left[\Lambda_n + \frac{\Gamma}{\zeta(1+\zeta/c)}\right] v_n = 0, \tag{4.5}$$

where $\Lambda_n \equiv gH^2S_z/f^2$, $\chi \equiv (gH^2S_z/f^2)(\beta/U)$, and $\Gamma \equiv \chi - 2/c$. The three parameters defining the drift, $\mathscr{U}$, z_0, c as well as S_z are regarded as arbitrary functions of y. The level z_0 may lie within or without the interval of interest.

To relate to previous results, the case of constant potential vorticity (k) may be distinguished in the present context. As $x \to \infty$ (2.7) transformed to physical space becomes $S_z = k/f$, or $S_{zy} = -k\beta/f^2$. Upon insertion of this expression into the curvature of the drift obtained from differentiation of (1.9*b*), there results $U_{zz} = g\beta k/f^3$; $U_{zz} = 2\mathscr{U}/H^2c$, whence $\chi = 2/c$ or $\Gamma = 0$.

4.2. *The westward drift condition*

To determine the lowest eigenvalue Λ_0 of (4.5) a variety of calculations have been made for various ranges or values of the parameters. Some exact solutions have been found, and variational and perturbation techniques have been applied. Λ_0 has always been found to be real, corresponding to pure real or imaginary λ_0. However, the physical results obtained can be summarized by merely considering the solution obtained by direct power series expansion about the regular singular point, $\zeta = 0$. Good accuracy can be expected for the lowest eigenfunction v_0. The solution for $\zeta > 0$ can be written (with subscripts suppressed)

$$v = \sum_{i=0}^{\infty} (a_i + b_i \ln \zeta)\,\zeta^i, \tag{4.6}$$

where
$$a_2 = -\frac{1}{2}\left\{\left[\frac{3}{2}\Gamma\left(1-\frac{\Gamma}{c}\right)+\Lambda\right]a_0 + \Gamma a_1\right\},$$

$$a_3 = \frac{1}{6}\left\{\Gamma\left(\frac{7}{6}\Gamma^2 + \frac{\Gamma}{3c} - \frac{1}{c^2} - \frac{1}{3}\Lambda\right)a_0 + \left[\Gamma\left(\frac{\Gamma}{2}+\frac{1}{c}\right)-\Lambda\right]a_1\right\}, \text{ etc.}$$

a_0 and a_1 are arbitrary, a_2 and a_3 have been given explicitly to expose the coupling of Λ with a_0 and a_1, and the b_i are related linearly to a_0. Convergence is indicated for $\zeta < |c| > 1$, $|\Gamma| < 1$.

It is instructive first to consider $z_0 = 0$, so that the open ocean drift is purely baroclinic, i.e. it is zero on the sea-bottom and is unidirectional. Here $a_0 = b_i = 0$ and a first approximation is obtained by satisfying the sea-surface condition with $a_{n\geqslant 4} = 0$. The result is

$$\Lambda = \frac{\chi^2}{2} - \chi\left[\frac{3}{2}+\frac{1}{c}\right], \quad 2c \gg 1. \tag{4.7}$$

Note that χ has the sign of $\mathscr{U}$ (which is also the sign of U). If $\beta = \chi = 0$, $\Lambda = 0$; there is no directionality condition on U but, since the lowest eigenfunction is x-independent, a possible modification of the flow at ∞ is indicated. Further interpretation cannot be made without an evaluation of the A_i. If χ differs slightly from zero Λ has sign opposite to the sign of χ, since $\frac{3}{2} > 1/c$ always. The conditions $|\Gamma| < 1$, $|c| > 1$ imply $\chi < 3$ so that the χ^2 term can never dominate (4.7). Thus $\mathscr{U} < 0$ or a westward geostrophic drift is always required for the existence of a boundary-layer. It is important to note that the case of constant potential vorticity, $\Gamma = 0$, is imbedded in these results (rather than being a special or limiting case). Thus the conclusions of § 3.3 may be anticipated to be generalizable. To check the accuracy of the qualitative results of (4.7) a numerical calculation for some typical values of Λ has been made to a higher order of approximation (for $a_4 \neq 0$). An accuracy of about 15 % is indicated for (4.7).

The westward condition on unidirectional flow does not appear to be particular to the forms of U, S which led from (4.3) to (4.5). As first pointed out to me by Mr S. L. Spiegel, the separated form of (4.1) may be integrated directly over the depth interval to yield

$$\lambda_n^2 = -\beta \int_0^H v_n \, dz \bigg/ \int_0^H U v_n \, dz. \tag{4.8}$$

Now it is usually the case that v_0 will have no node in the interval, thus $U < 0$ for $\lambda_0^2 > 0$ if U is of one sign. If this plausible result is accepted, then the case that U change sign in the interval becomes of great interest, i.e. is westward flow allowed at any level?

To explore the case that U has a single zero within the interval, choose $0 < z_0 < 1$ and let $q \equiv (1-z_0)/H$, $p \equiv 1-q$. An expansion valid for $\zeta < 0$ obtains similar to that of (4.6) by taking $\ln|\zeta|$; the two forms of the solution are joined across $\zeta = 0$ by the requirement that the vertical velocity w be continuous. If terms $O(\zeta^4)$ are retained in both directions, after satisfaction of the boundary conditions at $\zeta = q, -p$ there results the characteristic equation for the two lowest eigenvalues

$$Q(p,q;\chi,\Gamma)\,\Lambda^2 + R(p,q;\chi,\Gamma)\,\Lambda + T(p,q;\chi,\Gamma) = 0, \tag{4.9}$$

$$Q = \tfrac{1}{6}p^2q^2(p+q) + \tfrac{1}{9}p^2q^2\Gamma(\ln p - \ln q),$$

$$R = \tfrac{1}{3}(p^3+q^3) + \chi\{\tfrac{1}{3}pq(p^2-q^2) + [\tfrac{1}{6}p^2q^2(p-q)(\ln p - \ln q) + \tfrac{1}{36}p^2q^2(q+p)]\,\Gamma + \tfrac{1}{9}p^2q^2(\ln q - \ln p)\,\Gamma^2\}.$$

An investigation has been made of the signs of Q, R over the allowed ranges of p, Γ, c. It is found that $R/Q > 0$ always. Since this ratio is the negative of the sum of the roots of (4.9), $\Lambda_0 < 0$ if Λ_0 real. Numerical calculation of $\Lambda_0(\Gamma, c)$ for $q = 0{\cdot}1, 0{\cdot}5, 0{\cdot}9$ have been performed and real, negative Λ_0 found in every case.† It is concluded, therefore, that eastward geostrophic drift is entirely forbidden, everywhere within the depth interval if an inertial boundary current is to exist.

Care must be exercised in the interpretation of the results obtained here in the theory of the general ocean circulation. Solutions which are compounded of a geostrophic drift plus boundary induced flow which is not entirely confined to

† In our previous calculations (Robinson 1963, p. 160) for the case $q = 0{\cdot}5$, $\chi = 0$, the lowest eigenvalue was missed and it was incorrectly concluded that a boundary current could exist.

the boundary regions (one or a few eigenfunctions may oscillate and the others exponentially decay) may play a significant role in the total circulation.

Acknowledgement is made of the kind support of this work by both the Research Corporation and the National Science Foundation (Grant G 10323). It is a pleasure to thank Mr S. L. Spiegel for his assistance in the calculations, especially those of § 4.

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